Pitowsky Spacetimes, Malament-Hogarth Spacetimes, and All That

1. Supertasks

Trivial vs. non-trivial supertasks.

Relativity theory seems to place a limit on interesting supertasks. E.g. imagine an ∞-machine which tries to complete an infinite number of operations in a finite time by speeding up–it performs the first operation in 1 sec, the send in 1/2, sec, the third in 1/4 sec. etc. But to do this the parts of the machine must move faster and faster, and at some juncture they be moving faster than light, contradicting relativity theory. (Shrink the parts? Does this run into quantum limitations?)

On the other hand, relativistic spacetimes seem to open the possibility of performing bifurcated supertasks in which one party (the slave) has an infinite amount of proper time available in which to carry out an infinite number of operations, and the other party (the master) reaps the benefit of these labors.

2. Pitowsky spacetimes

> Def. 1 M, g_{ab} is a Pitowsky spacetime iff there are timelike half-curves $\gamma_1, \gamma_2 \subset M$ such that $\int_{\gamma_1} d\tau = \infty$, $\int_{\gamma_2} d\tau < \infty$, and $\gamma_1 \subset I^-(\gamma_2)$. (A timelike half-curve is a future-directed timelike curve which has past endpoint and which is inextendible in the future. For such a curve γ, $I^-(\gamma) := \cup_{p\in\gamma} I^-(p)$.)

Fact: Minkowski spacetime is Pitowskian.)

(a) See Fig. 1. γ_1 is an inertial trajectory and γ_2 spirals ever more tightly about γ_1 in order to have a finite proper length.

(b) See Fig. 2. γ_1 is a timelike curve with constant acceleration. $\int_{\gamma_1} d\tau = \infty$.

$\gamma_2 : u(t) = [1 - \exp(-2t)]^{1/2}$. $\int_{\gamma_2} d\tau = 1$. Acceleration of γ_2 is unbounded: the magnitude of the four-vector acceleration is $a(t) = \exp(2t)/[1 - \exp(-2t)]^{1/2}$, which blows up as $t \to \infty$.

Conjecture: Any spacetime that contains timelike half-curves of infinite proper length is Pitowskian.

Problems with trying to use Pitowsky spacetimes to perform bifurcated supertasks: 1) γ_2 has to satisfy physically unreasonable demands. 2) At no definite time does γ_2 know the fruits of γ_1's labors.

3. Malament-Hogarth spacetimes

> Def. 2 M, g_{ab} is a Malament-Hogarth spacetime iff there is a timelike half-curve $\gamma_1 \subset M$ such that $\int_{\gamma_1} d\tau = \infty$ and a point $p \in M$ such that $\gamma_1 \subset I^-(p)$.

Examples:

(a) Let M, g_{ab} be any spacetime. There is a real-valued $\Omega : M \to \mathbb{R}$ and a $p \in M$ such that $M - p, \Omega^2 g_{ab}$ is a Malament-Hogarth spacetime. Choose Ω which is unity outside some compact set C and which approaches ∞ as $p \in C$ is approached.

(b) Some spacetimes with CTCs will be Malament-Hogarth spacetimes because there are points p such that $I^-(p) = M$ and M contains timelike half-curves of infinite proper length. Examples: 2-dim Minkowski spacetime rolled up along the time axis; Gödel spacetime.

(c) anti-De Sitter spacetime. This spacetime is stably causal and, therefore, possess a global time function. See Fig. 3.

(d) Reissner-Nordstrom spacetime. See Fig. 4.

4. Properties of Malament-Hogarth spacetimes

(a) Malament-Hogarth spacetimes are not globally hyperbolic (i.e. do not possess a Cauchy surface). *Proof 1*: If M, g_{ab} is globally hyperbolic and $p, q \in M$ are timelike related, then there is a longest timelike curve connecting them. But in a M-H spacetime with a point $p \in M$ such that $\gamma_1 \subset I^-(p)$ and $\int_{\gamma_1} d\tau = \infty$, we can always beat any finite bound on the length of a timelike curve from the start point q of γ_1 to p. *Proof 2*: Suppose that the spacetime is globally hyperbolic and Malament-Hogarth. Choose a Cauchy surface S through p. γ_1 must intersect S at some point p'. But $\gamma_1 \subset I^-(p)$, so there is a timelike curve from p to p', and S is not achronal.

Thus, Minkowski spacetime and many other familiar spacetimes such as the FRW cosmological models are not Malament-Hogarth spacetimes. But Malament-Hogarth spacetimes are among the solutions to Einstein's field equations with sources satisfying standard energy conditions.

(b) Malament-Hogarth spacetimes involve divergent blue-shift effects. Suppose that M, g_{ab} contains a timelike half-curve $\gamma_1 \subset M$ such that $\int_{\gamma_1} d\tau =$

∞ and a point $p \in M$ such that $\gamma_1 \subset I^-(p)$. Suppose also that there is another timelike curve γ_2 from the start point q to p such that $\int_{\gamma_2} d\tau < \infty$. And suppose that the family of null geodesics from γ_1 to γ_2 forms a 2-dim sub-manifold in which the time order of emission matches the time order of reception (see Fig. 5). Let $\omega_1(\tau_1)$ and $\omega_2(\tau_2)$ be respectively the frequency of a photon at emission at γ_1 and the frequency at reception at γ_2. If $\omega_1(\tau_1)$ is constant along γ_1, then then $\int\limits_q^{p_2} \omega_2(\tau_2) d\tau_2$ diverges as $p_2 \to p$. This means that one can choose a sequence of points along γ_2 such that the frequency of received photons diverges as p is approached. It does not mean that every sequence will exhibit the divergent behavior.